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LETTER TO THE EDITOR

# On the remark of the existence of Virasoro symmetry for the non-linear $\sigma$ model

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**Abstract.** In this letter, we attempt to discuss the existence of the Virasoro symmetry, previously found by us, without two auxiliary constants introduced in the Hauser-Ernst linearisation equations for the O(3) non-linear $\sigma$ model.

The main objective of this letter is to state that the Virasoro symmetry for the non-linear $\sigma$ model exists even though one does not introduce two auxiliary constants $\alpha$ and $\beta$ which arise in the Hauser-Ernst linearisation equations [1]. Our work extends our results [1, 2] which recognise the existence of new kinds of Kac-Moody symmetry and Virasoro symmetry for the non-linear $\sigma$ model and the Ernst equation. More importantly, it provides an effective approach to the discovery of the similar Virasoro symmetry in the solution space of the principal chiral model, which will be presented in a future paper.

In [1], we found that the O(3) non-linear $\sigma$ model is more conveniently described in the similar Hauser-Ernst formalism [3] rather than the usual Zakharov-Mikhailov formalism [4]. For our description, the equation of the basic field $N \in \mathrm{O}(3)$

$$\mathrm{d}(N^* \mathrm{d}N) = 0 \qquad N^2 = \alpha^2 \tag{1}$$

can be written into the equivalent form, the matrix Ernst equation

$$2(\beta + \alpha^*)\, \mathrm{d}E = (E + E^+)\, \mathrm{d}E \tag{2}$$

where $E$ is the $2 \times 2$ matrix Ernst potential

$$E = N + X \qquad \operatorname{tr} E = 2\beta \tag{3}$$

with the matrix twist potential $x$, and $\alpha$, $\beta$ are constants. Here we use the differentiation form and the asterisk denotes the dual operation in the light-cone coordinates

$$\xi = \tfrac{1}{2}(x^1 + x^2) \qquad \eta = \tfrac{1}{2}(x^1 - x^2). \tag{4}$$

Then we establish the Hauser-Ernst linearisation equations and the subsidiary conditions.

Using solutions to the linearisation equations we propose two infinitesimal transformations and show that they can keep (2) invariant and constitute the Virasoro and Kac-Moody algebras. In particular, we point out that $\alpha$ and $\beta$ play an imporant role in maintaining this Virasoro symmetry. However, we feel that the constants $\alpha$ and $\beta$ should not have to be changed under the transformation. The question is whether the

Virasoro symmetry still holds or not if we change $\alpha$ and $\beta$. After we succeeded in developing a new infinitesimal transformation we realised that the existence of the Virasoro symmetry for the non-linear $\sigma$ model does not depend on $\alpha$ and $\beta$. The present letter is an attempt to discuss this.

As is usual [5], we take

$$\alpha = 1 \qquad \beta = 0. \tag{5}$$

Then (2) reduces to

$$2\,{*}\mathrm{d}E = (E + E^{+})\,\mathrm{d}E \tag{6}$$

or, equivalently,

$$2\,\mathrm{d}E = A(t)\Gamma(t) \tag{7}$$

where

$$A(t) = I - t(E + E^{+}) \tag{8}$$

$$\Gamma(t) = t[1 - 2t^{*}]^{-1}\,\mathrm{d}E. \tag{9}$$

The corresponding linearisation equations with the auxiliary condition are written as

$$\mathrm{d}F(t) = \Gamma(t)F(t) \tag{10}$$

$$F(0) = I \tag{11}$$

$$\dot{F}(0) = E \tag{12}$$

$$F(t)^{\times}A(t)F(t) = I \tag{13}$$

$$\det F(t) = 1 \tag{14}$$

where $\dot{F}(t) = (\partial/\partial t)F(t)$, $F(t)^{\times} = F^{+}(\bar{t})$ (where the bar denotes the complex conjugation).

As mentioned before, we cannot use our previous Virasoro symmetry transformation to treat the invariance of (6). We thus present here another form of the infinitesimal transformation

$$\delta E = -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{[1-(2x)^2]^2}{x^4(x-s)}\dot{F}(x)F(x)^{-1}\,\mathrm{d}x \tag{15}$$

where an infinitesimal constant is not written out, and $C_{0,s}$ denotes a circle surrounding $x = 0, s$ in the complex parameter $x$ plane, and $F(x)$ is a solution to (10)-(14). We can also prove that (6) will keep invariant under the new infinitesimal transformation up to $\delta^2$, i.e.

$$2\,{*}\mathrm{d}\delta\,E = (E + E^{+})\,\mathrm{d}\delta\,E + (\delta E + \delta E^{+})\,\mathrm{d}E. \tag{16}$$

To prove this, differentiate (15) and use (10) to get

$$\begin{aligned}
\mathrm{d}\delta E &= -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{[1-(2x)^2]^2}{x^4(x-s)}\,\mathrm{d}(\dot{F}(x)F(x)^{-1})\,\mathrm{d}x \\
&= -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{[1-(2x)^2]^2}{x^4(x-s)}\{\dot{\Gamma}(x) + [\Gamma(x), \dot{F}(x)F(x)^{-1}]\}\,\mathrm{d}s \\
&= -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{[1-(2x)^2]^2}{x^4(x-s)}[\Gamma(x), \dot{F}(x)F(x)^{-1}]\,\mathrm{d}s
\end{aligned} \tag{17}$$

where we have used

$$\int_{C_{0,s}} \frac{\mathrm{d}s}{x^{k+1}(x-s)} = 0 \qquad k \geqslant 0. \tag{18}$$

Since $^*\Gamma(x) = (E+E^+)\Gamma(x)$ from (6) and (9), using (17)

$$\begin{aligned}
^*\mathrm{d}\delta E &= -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{[1-(2x)^2]^2}{x^4(x-s)}[^*\Gamma(x), \dot{F}(x)F(x)^{-1}]\,\mathrm{d}x \\
&= -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{[1-(2x)^2]^2}{x^4(x-s)}\{[E+E^+, \dot{F}(x)F(x)^{-1}]\Gamma(x) \\
&\quad +(E+E^+)[\Gamma(x), \dot{F}(x)F(x)^{-1}]\}\,\mathrm{d}x \\
&= (E+E^+)\,\mathrm{d}\delta E - \frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{[1-(2x)^2]^2}{x^4(x-s)}[E+E^+, \dot{F}(x)F(x)^{-1}]\,\mathrm{d}x.
\end{aligned} \tag{19}$$

On the other hand, using (7), (13) and (18):

$$\begin{aligned}
&(\delta E+\delta E^+)\,\mathrm{d}E \\
&\quad = -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{[1-(2x)^2]^2}{x^4(x-s)}\{\dot{F}(x)F(x)^{-1}+(F(x)^\times)^{-1}\dot{F}(x)^\times\}\frac{1}{x}A(x)\Gamma(x)\,\mathrm{d}x \\
&\quad = -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{[1-(2x)^2]^2}{x^5(x-s)}\{[-A(x), \dot{F}(x)F(x)^{-1}]-\dot{A}(x)\}\Gamma(x)\,\mathrm{d}x \\
&\quad = -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{(1-(2x)^2)^2}{x^5(x-s)}\{x[E+E^+, \dot{F}(x)F(x)^{-1}]+(E+E^+)\}\Gamma(x)\,\mathrm{d}x \\
&\quad = -\frac{1}{2\pi\mathrm{i}}\int_{C_{0,s}} \frac{(1-(2x)^2)^2}{x^4(x-s)}[E+E^+, \dot{F}(x)F(x)^{-1}]\,\mathrm{d}x.
\end{aligned} \tag{20}$$

Finally, substitute (20) into (19) to complete the proof of (16).

We can also prove that conditions (5) do not vary under the transformation (15). Since from (14) and $\mathrm{tr}(\dot{F}(t)F^{-1}(t)) = (\det F(t))^{-1}(\partial/\partial t)\det F(t)$, it follows

$$\delta\,\mathrm{tr}\,E = -\frac{1}{2\pi\mathrm{i}}\int \frac{[1-(2x)^2]^2}{x^4(x-s)}\mathrm{tr}(\dot{F}(t)F(t)^{-1}) = 0 \tag{21}$$

and also

$$\begin{aligned}
\delta(\det N) &= \tfrac{1}{2}\mathrm{tr}(N\delta N) \\
&= \tfrac{1}{8}\mathrm{tr}[(E+E^+)(\delta E+\delta E^+)] \\
&= \tfrac{1}{8}\mathrm{tr}\left[\left(\frac{1-A(x)}{x}\right)(\delta E+\delta E^+)\right] \\
&= -\frac{1}{8x}\mathrm{tr}[A(x)(\delta E+\delta E^+)] \\
&= -\frac{1}{8x}\mathrm{tr}\,E = 0
\end{aligned} \tag{22}$$

where we have used (13) and (21).

In order to investigate the algebra structure of the new symmetry transformation we give the corresponding transform of $F(t)$ as

$$\delta F(t) = -\frac{t}{2\pi\mathrm{i}}\int_{C_{0,s,t}} \frac{(1-(2x)^2)^2}{x^3(x-t)(x-s)}\dot{F}(x)F(x)^{-1}\,\mathrm{d}x\,F(t) \tag{23}$$

due to (15). Expanding (23) in powers of $s$, we get the following expression:

$$\delta^{(k)}F(t)=(L_k+8L_{k+2}+16L_{k+4})F(t) \tag{24}$$

where $L_k$ $(k\geqslant 0)$ is defined as

$$L_kF(t)=-\frac{t}{2\pi i}\int_{C_{0,s,t}}\frac{x^{-k}}{x-s}\dot{F}(x)F(x)^{-1}\,dx\,F(t) \tag{25}$$

which constitute the half Virasoro algebra [1], i.e.

$$[L_k, L_l]F(t)=(k-l)L_{k+l}F(t). \tag{26}$$

Therefore, it is easy to evaluate the following commutations:

$$[\delta^{(k)}, \delta^{(l)}]F(t)=(k-l)\{\delta^{(k+l)}+8\delta^{(k+l+2)}+16\delta^{(k+l+4)}\}F(t). \tag{27}$$

So our new infinitesimal symmetry transformations possess an infinite-dimensional conformal algebra. In terms of our recent work [6], all the infinite-dimensional conformal symmetries, involving the present and the old [1], are originated from the generalised Riemann-Hilbert transformation.

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